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Ramsey Varieties

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The Ramsey problem is considered for various classes of (universal) algebras. It is shown that every variety has singleton-Ramsey property and examples of varieties with singleton-Ramsey property only are given.

1. INTRODUCTION

The following scheme was abstracted from the Ramsey type results by K. Leeb (see [4] and e.g. [6]):

Let $\mathcal{K}$ be a category. A subobject of an object $B \in \mathcal{K}$ is a class of morphisms $[f] = \{f \circ \varphi;$ φ an automorphism of $A\}$ for an $f: A \to B$. If A, B are two objects of $\mathcal{K}$, then $\binom{B}{A}$ denotes the class of all subobjects of B which are isomorphic to A.

$\mathcal{K}$ is said to have A-Ramsey property if for every $B \in \mathcal{K}$ there exists a $C \in \mathcal{K}$ such that for every partition $\binom{C}{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ there exists an $i \in \{1, 2\}$ and a morphism $g: B \to C$ such that $[g \circ f] \in \mathcal{A}_i$ for every $f \in \binom{B}{A}$. In this case we say that C is A-Ramsey for B.

$\mathcal{K}$ is said to be Ramsey if $\mathcal{K}$ has A-Ramsey property for every $A \in \mathcal{K}$.

Ramsey problem for $\mathcal{K}$ is the determination of the class of all $A \in \mathcal{K}$ for which $\mathcal{K}$ has A-Ramsey property. This class will be denoted by $r(\mathcal{K})$.

The Ramsey problem was solved for several combinatorial and algebraical categories:
(*a*) the category of all finite sets and all 1–1 mappings [10];
(*b*) the category of all finite graphs and all embeddings [5];
(*c*) the category of all finite vector spaces [2];
(*d*) the category of all finite abelian groups [11].

This note was motivated by the investigation of varieties of lattices. Extending results of [7] and answering a question of [8] we prove in Section 2 that every variety of finite lattices has singleton Ramsey property (i.e. A-Ramsey property for $|A| = 1$).

It seems that the category of all finite modular lattices has singleton Ramsey property only. In Section 3 we exhibit examples of varieties of groupoids which have singleton Ramsey property only.

We use two different techniques: in Section 2 we apply the Hales–Jewett theorem to classes of algebras which are closed for products (e.g. varieties). This application of Hales–Jewett theorem was realized, independently, by B. Voigt and H. J. Prömmel (Bielefeld). In Section 3 we use amalgamation techniques to classes of algebras "without many identities".

2. PRODUCTS

A type Δ is a set of symbols, each of which is associated with a non-negative integer, called its arity. A Δ-algebra A is a pair $(X, (F_A)_{F \in \Delta})$ where X is a non-empty set and F_A is an n-ary operation on X whenever $F \in \Delta$ is n-ary. The set X (the underlying set of A) will be identified with A.

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Let A, B be two Δ-algebras. A monomorphism $f: A \to B$ is a 1–1 mapping satisfying

$$f(F_A(x_1, \dots, x_n)) = F_B(f(x_1), \dots, f(x_n))$$

for every choice of $F \in \Delta$ and $x_1, \dots, x_n \in A$; n is the arity of F.

A subalgebra of an algebra A is a subset B which is closed for the operations of A; the operations of B are the restrictions of the operations of A. If A, B are two Δ-algebras, then $\binom{A}{B}$ denotes the set of all subalgebras of A which are isomorphic to B.

Let A, B be Δ-algebras. We define the product $C = A \times B$ as follows: its underlying set is the cartesian product of the underlying sets of A, B;

$$F_C((a_1, b_1), \dots, (a_n, b_n)) = (F_C(a_1, \dots, a_n), F_C(b_1, \dots, b_n)).$$

The product of an arbitrary family of Δ-algebras is defined analogously.

A class $\mathcal{K}$ of Δ-algebras will be considered as a category with respect to monomorphisms. Notice that subobjects in this category coincide with subalgebras. Using this, if $A \in \mathcal{K}$, then $\mathcal{K}$ has A-Ramsey property iff for every $B \in \mathcal{K}$ there exists a $C \in \mathcal{K}$ such that for every partition $\binom{C}{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ there exists an $i \in \{1, 2\}$ and an algebra $B' \in \binom{C}{B}$ with $\binom{B'}{A} \subseteq \mathcal{A}_i$.

Proposition 2.1. Let $\mathcal{K}$ be a class of finite Δ-algebras which is closed for products and subalgebras. Then $\mathcal{K}$ has $\mathbb{1}$-Ramsey property, where $\mathbb{1}$ is the singleton Δ-algebra.

In the proof we shall use the following well known theorem (see e.g. [3]): An integer $n > 0$ is identified with the set $\{0, 1, \dots, n-1\}$. For sets A and B let ${}^AB$ be the set of all mappings $A \to B$.

Hales–Jewett Theorem. *For every k, n there exists an N with the following property: For every partition ${}^Nk = \mathcal{A}_1 \cup \mathcal{A}_2$ there exists a subset ω of N, a function $f_0 \in {}^Nk$ and an $i \in \{1, 2\}$ such that $f \in \mathcal{A}_i$ for every mapping $f \in {}^Nk$ which satisfies*

$$f(j) = f_0(j), \text{ for } j \notin \omega, \qquad f|_\omega = \text{constant}.$$

Proof of Proposition 2.1. Let $B \in \mathcal{K}$. Denote by X_0 the set of all those $x \in B$ for which the set $\{x\}$ is a subalgebra of B. Put $|B| = a$, $|X_0| = k$. Without loss of generality assume $X_0 = k$. Let N be the number guaranteed by the Hales–Jewett theorem. Put $C = B^N = B \times \dots \times B$. We shall prove that C is $\mathbb{1}$-Ramsey for B.

Consider $\binom{C}{\mathbb{1}}$. If $Z \in \binom{C}{\mathbb{1}}$ then $|Z| = 1$, $Z = \{(z_i)_{i \in N}\}$ and obviously $z_i \in X_0$ for every i. Thus the singleton subalgebra Z may be identified with the element $(z_i)_{i \in N}$ and the set $\binom{C}{\mathbb{1}}$ may be identified with the set ${}^Nk$. Consequently, given a partition $\binom{C}{\mathbb{1}} = \mathcal{A}_1 \cup \mathcal{A}_2$, we can define a partition ${}^Nk = \mathcal{B}_1 \cup \mathcal{B}_2$. Using the Hales–Jewett property of N there exist $\omega \subseteq N$, a function f_0 and an $i \in \{1, 2\}$ such that the set H_0 of all functions $f \in {}^Nk$ which are constant on ω and which coincide with f_0 on $N \backslash \omega$ belongs to $\mathcal{B}_i$. Extend H_0 to the set of all functions $f \in {}^Na$ which are constant on ω and which coincide with f_0 on $N \backslash \omega$. Obviously the subalgebra B' of C induced by this set is isomorphic to B. Moreover $\binom{B'}{\mathbb{1}} = H_0 \subseteq \mathcal{A}_i$.

COROLLARY 2.2. (cf. [8]). *Any variety of finite lattices has* $\mathbb{1}$*-Ramsey property.*

3. AMALGAMATIONS

A set M of sets is said to be simple if $X, Y \in M$ and $X \neq Y$ imply $|X \cap Y| \leq 1$.

LEMMA 3.1. *For every $n \geq 1$ there exists a set M of n-element sets such that*
(1) *M is simple;*
(2) *if $\bigcup M = P \cup Q$ then there exists an $X \in M$ with either $X \subseteq P$ or $X \subseteq Q$.*

PROOF. See [1].

PROPOSITION 3.2. *Let Δ be a type and $\mathscr{K}$ be an abstract (i.e. closed for isomorphic objects) class of finite Δ-algebras such that for any finite consistent family J_X $(X \in M)$ of algebras from $\mathscr{K}$ there exists an algebra $C \in \mathscr{K}$ such that J_X is a subalgebra of C for any $X \in M$. (A family J_X is said to be consistent if for any $X, Y \in M$ such that $J_X \cap J_Y \neq \emptyset$ there exists an algebra H such that H is a subalgebra of both J_X, J_Y and the underlying set of H coincides with $J_X \cap J_Y$.) Let $A \in \mathscr{K}$ be an algebra without proper subalgebras. Then $A \in r(\mathscr{K})$.*

PROOF. Let $B \in \mathscr{K}$. Put $n = \left|\binom{B}{A}\right|$ and $\binom{B}{A} = \{A_1, \ldots, A_n\}$, so that $A_1, \ldots, A_n$ are pairwise disjoint subalgebras of B isomorphic to A. For every $i = 1, \ldots, n$ let α_i be an isomorphisms of A onto A_i. Let M be as in 3.1. For every $X \in M$ fix an ordering $o_1^X, \ldots, o_n^X$ of elements of X and put

$$J_X = \{(o_i^X, a);\, o_i^X \in X, a \in A\} \cup \{(X, b);\, b \in B \backslash (A_1 \cup \ldots \cup A_n)\}.$$

For every $X \in M$ define a 1–1 mapping γ_X of J_X onto B as follows: $\gamma_X(o_i^X, a) = \alpha_i(a)$; $\gamma_X(X, b) = b$. There exists a unique algebra with the underlying set J_X (we shall denote it by J_X, too) such that γ_X is an isomorphism of J_X onto B. Evidently, if $F \in \Delta$ is a symbol of arity k, if $X \in M$, $o_i^X \in X$ and $a_1, \ldots, a_k \in A$ then

$$F_{J_X}((o_i^X, a_1), \ldots, (o_i^X, a_k)) = (o_i^X, F_A(a_1, \ldots, a_k)).$$

From this it follows that the family J_X $(X \in M)$ is consistent and so there exists an algebra $C \in \mathscr{K}$ such that J_X is a subalgebra of C for any $X \in M$. Let $\binom{C}{A} = \mathscr{A}_1 \cup \mathscr{A}_2$. Denote by P the set of all $o \in \bigcup M$ such that the subalgebra $\{(o, a);\, a \in A\}$ of C belongs to $\mathscr{A}_1$ and by Q the set of all $o \in \bigcup M$ such that $\{(o, a);\, a \in A\} \in \mathscr{A}_2$. Evidently, $\bigcup M = P \cup Q$. Hence there exists an $X \in M$ with either $X \subseteq P$ or $X \subseteq Q$. We have $J_X \in \binom{C}{B}$ and either $\binom{J_X}{A} \subseteq \mathscr{A}_1$ or $\binom{J_X}{A} \subseteq \mathscr{A}_2$.

COROLLARY 3.3. *Let $\mathscr{K}$ be the class of all finite groupoids. Then every finite groupoid without proper subgroupoids belongs to $r(\mathscr{K})$.*

COROLLARY 3.4. *Let $\mathscr{K}$ be the class of all finite commutative groupoids. Then every finite commutative groupoid without proper subgroupoids belongs to $r(\mathscr{K})$.*

LEMMA 3.5. *For every $n \geq 1$ there exist a finite set M of n-element sets and for every $X \in M$ a linear ordering r_X of X such that the following conditions are satisfied:*

(1) *M is simple;*

(2) *for every linear ordering r of $\bigcup M$ there exists an $X \in M$ with $r|_X = r_X$.*

PROOF. See [9].

An algebra is said to be idempotent if any element is a subalgebra. An algebra is said to be homogeneous if any permutation is an automorphism.

PROPOSITION 3.6. *Let $\mathcal{K}$ be an abstract class of finite idempotent Δ-algebras satisfying the following condition:*

(A1) *If M is a finite simple set of algebras from $\mathcal{K}$ then there exists an algebra $B \in \mathcal{K}$ such that all the algebras from M are subalgebras of B.*

Let $A \in r(\mathcal{K})$. Then A is a homogeneous algebra.

PROOF. Suppose that a permutation p of A is not an automorphism of A. Put $A = \{a_1, \dots, a_n\}$, $p(a_i) = a_{p(i)}$. Let $a'_1, \dots, a'_n$ be pairwise different elements not belonging to A and denote by A' the algebra with the underlying set $\{a'_1, \dots, a'_n\}$ such that $a_i \mapsto a'_i$ is an isomorphism of A onto A'. By (A1) there is an algebra $A^* \in \mathcal{K}$ such that A, A' are subalgebras of A^*. By Lemma 3.5 there exists a finite set M of sets of cardinality $|A^*|$ and for every $X \in M$ a linear ordering r_X of X such that M is simple and for every linear ordering r of $\bigcup M$ there is an $X \in M$ with $r|_X = r_X$. Define a linear ordering s of A^* so that $(a_i, a'_j) \in s$ for all i, j, $(a_i, a_j) \in s$ iff $i \leq j$ and $(a'_i, a'_j) \in s$ iff $p(i) \leq p(j)$. For every $X \in M$ denote by $\bar{X}$ the unique algebra with the underlying set X such that the unique order-isomorphism o_X of (A^*, s) onto (X, r_X) is an isomorphism of the algebra A^* onto $\bar{X}$. By (A1) there exists an algebra $B \in \mathcal{K}$ such that all $\bar{X}$ are subalgebras of B. Let $C \in \mathcal{K}$. Let us fix a linear ordering r of C. Denote by $\mathcal{A}_1$ the set of all $A' \in \binom{C}{A}$ such that the unique order-isomorphism of (A, s) onto (A', r) is an isomorphism of the algebra A onto A'. Put $\mathcal{A}_2 = \binom{C}{A} \setminus H_1$. Let $B' \in \binom{C}{B}$; let φ be an isomorphism of B onto B'. There is an $X \in M$ with $r_X = \varphi^{-1}(r|_{B'})|_X$. Evidently $\varphi(o_X(A)) \in \mathcal{A}_1$ and $\varphi(o_X(A')) \in \mathcal{A}_2$.

PROPOSITION 3.7. *Let V be a variety of finite idempotent groupoids determined by a set of identities in only two variables and containing a two-element groupoid. Let $A \in r(V)$. Then A is a homogeneous groupoid.*

PROOF. By 2.6, it is enough to prove (A1). Put $B = \bigcup M$ and define a multiplication on B as follows: if $a, b \in X$ for some $X \in M$, define ab in the same way as in X; if no, let $\{a, b\}$ be a two-element subgroupoid belonging to V. Since V is determined by identities in only two variables, $B \in V$.

PROPOSITION 3.8. *Let G be a groupoid. Then G is a homogeneous groupoid iff one of the following cases takes place:*

(1) *G satisfies $xy = x$;*

(2) *G satisfies $xy = y$;*

(3) *G is isomorphic to the groupoid with multiplication table*

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2 & 1 \\ 2 & 2 & 1 \end{array} \;;$$

(4) *G is isomorphic to the groupoid with multiplication table*

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 2 & 2 \\ 2 & 1 & 1 \end{array} \;;$$

(5) *G is isomorphic to the groupoid with multiplication table*

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{array} \;;$$

PROOF. It is self-evident.

EXAMPLE 3.9. Let V be the variety of finite groupoids determined by the identities

$$xy = yx, \qquad xx = x, \qquad x(xy) = xy.$$

Then $r(V)$ contains only the one-element groupoid.

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